111.

AN EXAMPLE OF THE METHOD OF DEDUCING A SURFACE FROM A PLANE FIGURE. [112]

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Let there be given, in a plane π, six (fundamental) points **1**, **2**, **3**, **4**, **5**, **6**, of which neither any three lie in a right line, nor all in a conic; and consider the six conics $[1] \equiv \mathbf{23456}$, $[2] \equiv \mathbf{13456}$, $[3] \equiv \mathbf{12456}$, $[4] \equiv \mathbf{12356}$, $[5] \equiv \mathbf{12346}$, $[6] \equiv \mathbf{12345}$, and the fifteen right lines $\overline{\mathbf{12}}$, $\overline{\mathbf{13}}$, .. , $\overline{\mathbf{16}}$, $\overline{\mathbf{23}}$, ... $\overline{\mathbf{56}}$.

There is a pencil of cubics $\mathbf{1^2 23456}$ (curves of the third order, having a node at **1** and passing through the other fundamental points); their tangents at the common node form an involution, viz., they are harmonically conjugate with regard to two fixed rays. Five pairs of conjugate rays of this involution are already known; for instance, the line $\overline{\mathbf{12}}$ and the conic [2] have conjugate directions at the point **1**, for, they make up a cubic $\mathbf{1^2 23456}$.

Each other fundamental point is the centre of a like involution. And also on each conic [1], [2], ..., and each line $\overline{\mathbf{12}}$, $\overline{\mathbf{13}}$, ... points are coupled harmonically with regard to two fixed points. The involution on the conic [1] is cut by the pencil of rays through **1**; for instance, the point **2** is conjugate to the second intersection of [1] with $\overline{\mathbf{12}}$, &c. The involution on the line $\overline{\mathbf{12}}$ is cut by the pencil of conics **3456**; for instance, the points $\overline{\mathbf{12}}\,.\,\overline{\mathbf{34}}$, $\overline{\mathbf{12}}\,.\,\overline{\mathbf{56}}$ are conjugate, as $\overline{\mathbf{34}}$ and $\overline{\mathbf{56}}$ make up a conic through **3456**; and the point **1** is conjugate to the second meeting of $\overline{\mathbf{12}}$ with the conic [2]; &c.

The Jacobian of a linear twofold system (réseau) of cubics **123456** is a sextic $K \equiv (\mathbf{123456})^2$ having six nodes at the fundamental points. Since any réseau of cubics **123456** contains 1° a cubic $k \equiv \mathbf{1^2 23456}$; 2° a cubic breaking up into a ray r through **1** and the conic [1]; 3° a cubic made up by the line $\overline{12}$ and a conic c trough **3456**, &c.; we see immediately that the (sextic K) Jacobian of the réseau 1° has the same tangents as the cubic k at the common node **1**; 2° and 3° passes through the intersections of r with **[1]**, and the intersections of c with $\overline{12}$ &c.

The Jacobians K form a linear threefold system of sextics $(\mathbf{123456})^2$

$$\lambda K + \lambda' K' + \lambda'' K'' + \lambda''' K''' = 0,$$

therefore we have the following theorem:

If six points **1, 2, 3, 4, 5, 6,** are given in a plane π, as said above, we may construct a threefold linear system of sextics $K \equiv (\mathbf{123456})^2$, whose tangents at each of the six common nodes are coupled in involution, and which cut, also in involution, each of the six conics **[1], [2]**,.. and of the fifteen right lines $\overline{12}$, $\overline{13}$, ... Any sextic of this system is the Jacobian of a réseau of cubics **123456.**

Among these ∞^3 cubics, there are ∞^1 curves possessing a cusp (stationary point), and the locus of the cusps is a curve $\Theta \equiv (\mathbf{123456})^4$ of the twelfth order, which touches each conic [1], [2],... and each line $\overline{12}$, $\overline{13}$, ... in two distinct points, and has (only) two distinct tangents at each quadruple point **1, 2**,...: those points and these tangents being the double elements of the twenty-seven involutions mentioned above.

Let us start now from the foregoing plane diagram, without any further reference to its origin; and consider π as representative of a surface Φ whose plane sections shall have the sextics K as their images. *) We see at once that the order of Φ is 12, for, two sextics K meet in $(6.6 - 6.4 =)$ 12 more points. Thus we get a (1, 1) correspondence between the points of π and those of Φ; any point M on π being common to ∞^2 sextics K, it is the image of a point M' on Φ, in which the ∞^2 corresponding planes meet. But if M lies on one of the six conics [1], [2],... or of the fifteen lines $\overline{12}$, $\overline{13}$,... or infinitely near to one of the six points **1, 2**,..., then all the ∞^2 sextics K passing through M contain also another common point M_1, which is conjugate to M in one of the twenty-seven involutions. Therefore, in such case, M' is a double point on Φ: this surface has an infinite range of double points, whose locus, as easy to see, is constituted by twenty-seven right lines, having as their images on π the six fundamental points and the six conics and fifteen lines connecting them.

If M falls at the intersection of $\overline{12}$, $\overline{34}$, viz., if it belongs to two involutions, it will

*) See CAPORALI's paper in *Collectanea Math. in memoriam D. Chelini.*

have two conjugate points $M_1 \equiv (\overline{12})(\overline{56})$, $M_2 \equiv (\overline{34})(\overline{56})$; and the three points M M_1 M_2 will be common to ∞^2 sextics K correspondig to ∞^2 planes, whose point of intersection M′ (where the nodal lines of Φ meet, which answer to $\overline{12}$, $\overline{34}$, $\overline{56}$) is consequently a treble point on Φ. Thus, our surface possesses forty-five treble points, in each of which three nodal lines meet.

Let a cubic **123456** have a cusp M; then, every sextic $(\mathbf{123456})^2$, which is the Jacobian of a réseau including that cubic, shall pass through M and touch there the tangent at the cusp. Hence the ∞^2 sextics K through M will have the same tangent at this point. Accordingly the corresponding point M′ will be a double point on Φ with coinciding tangent planes, viz., a cuspidal or stationary point. Thus we see that Φ has a cuspidal curve, whose image on π is the locus of cusps of cubics **123456**, viz., the curve $\Theta \equiv (\mathbf{123456})^4$ of the twelfth order. The order of the cuspidal curve on Φ is $(6.12 - 6.2.4 =) \, 24$.

The class of Φ, that is to say, the number of the tangent planes drawn through two arbitrary points in space, is equal to that of the intersections of the Jacobians of two linear twofold systems of sextics K. The Jacobian of such a system is of the order $3(6-1) = 15$, and passes $3.2 - 1 = 5$ times through each fundamental point; but the curve Θ is clearly included in the Jacobian, therefore, this latter will break up into a fixed curve, Θ, and a variable one, being of the order $15 - 12 = 3$, and possessing the multiplicity $5 - 4 = 1$ at the fundamental points. So the residual Jacobian is a cubic curve **123456**. Two such curves meet in $9 - 6 = 3$ more points; hence the class of Φ is 3.

The surface Φ, being of the twelfth order and third class, and having twenty-seven nodal right lines and a cuspidal curve of the twenty-fourth degree, is the reciprocal of the general cubic surface. It was very easy to foresee this conclusion, in accordance with the (1, 1) correspondence between any surface and its reciprocal. But I wished to give an instance of the method of deducing a (unicursal) surface from a plane figure assumed as its representative.