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ON THE FOURTEEN-POINTS CONIC. [77]

By prof. CREMONA.

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Theorem. If ω, ω' be the two points on any side of a complete quadrilateral, each of which determines, with the three vertices on that side, an equianharmonic system; and if i, i' be the double points of the involution determined, on any diagonal, by two opposite vertices and by the intersections of the other two diagonals; then the four pairs of points ω, ω' will lie, with the three pairs i, i', upon one and the same conic.

Demonstration. Let α, β, γ be the corners of the triangle formed by the diagonals which connect the opposite vertices a, a'; b, b'; c, c'; and on any side, say abc, let a point ω be taken so as to make the anharmonic ratio $(abc\omega)$ equal to one of the imaginary cube roots of -1. The four points ω, relative to the four triads abc, $ab'c'$, $a'bc'$, $a'b'c$, will be the points of contact of a conic Σ inscribed in the quadrilateral, since these points of contact necessarily determine homographic ranges and the diagonals aa', bb', cc' represent three of the inscribed conics. Similarly, if ω' be taken so as to make the anharmonic ratio $(abc\omega')$ equal to the *other* imaginary cube root of -1, the four points ω' will be points of contact of another inscribed conic Σ'.

Again, the eight points of contact of *any* two inscribed conics Σ and Σ' lie, as is well known, on a third conic S, with respect to which the triangle $\alpha\beta\gamma$ is self-conjugate; the polar of a relative to S, therefore, will pass through α. This polar will, moreover, pass through A, the harmonic conjugate of a relative to bc, since $\omega\omega'$ is divided harmonically by a and A, and, passing through α and A. It will necessarily also pass through the vertex a', opposite to a. But if so, the conic S, which is already known to cut $\beta\gamma$ harmonically, will do the same to aa', and consequently will pass through the points i, i'. By similar considerations with respect to the other two diagonals, therefore, the theorem may readily be established.